ON THE WEDGING OF BRITTLE BODIES

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This paper contains the formulation and an effective solution of the dynamic plane strain problem of wedging of brittle bodies by an infinite rigid wedge of arbitrary shape moving with constant velocity. The study is conducted on the basis of the general concepts on brittle cracking developed in papers [1] where, in particular, a problem of wedging in static formulation was investigated. Special cases are considered, namely a wedge of constant thickness and rounded wedges of various shape.

The limiting values of the wedge velocity are found, up to which the present formulation of the problem is valid. The special role played by Rayleigh waves in wedging problems, in problems of uniform motion of a rigid stamp on the surface of a semispace [2,3,4] and other analogous problems is investigated; it turns out that as the Rayleigh velocity is reached peculiar resonance phenomena are produced in the elastic body. It is shown that the velocity of formation of a free crack can never exceed the Rayleigh surface-wave velocity in the given material. The problem investigated is of particular interest in the theory of cutting, since cutting at high speeds is invariably accompanied by wedging in most cases follows the brittle or quasi-brittle mechanism.

1. Introduction and formulation of the problem. We consider a homogeneous and isotropic brittle body, subjected to wedging by an asymmetric, absolutely rigid wedge (Fig. 1), which is of thickness $2h$ at infinity and which moves along its line of symmetry with constant velocity V. Velocity V is assumed to be smaller than the shear-wave velocity c_2 in the wedged body.* A crack is formed in front of the wedge, which

* As the investigation below shows, the given formulation of the problem is possible only if the wedge velocity is smaller than the velocity of Rayleigh surface waves, which in turn is smaller than the velocity in the given material.

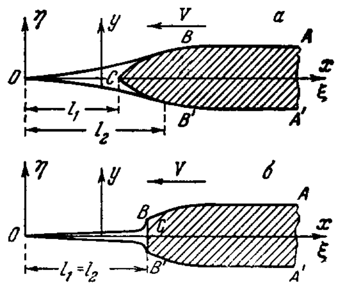

Fig. 1.

closes smoothly at a certain point O; the location of the point O with respect to the front point of the wedge C is not known in advance and must be determined in the process of solving the problem.

If the wedge has a rounded front part (Fig. 1a), then the location of beginning of contact between the crack surface and the wedge B and B' is also not known in advance and is determined in the process of solving the problem.

If the wedge possesses a truncated front part (Fig. 1b), as for example in the case of a wedge of constant thickness, then the location of the closing points is completely determined: they coincide with the corners of the front part of the wedge. However, in this case the stresses at the closing points are infinite. The friction forces, acting on the faces AB and $A'B'$ of the wedge which touch the wedged body, are assumed to follow Coulomb's law, with a coefficient of friction k.

The mathematical formulation of the problem is reduced to the following: the equations of dynamic plane elasticity

$$(\lambda + \mu)\frac{\partial\theta}{\partial x} + \mu\,\Delta u - \rho\frac{\partial^2 u}{\partial t^2} = 0 \qquad (1.1)$$

$$(\lambda + \mu)\frac{\partial\theta}{\partial y} + \mu\Delta v - \rho\frac{\partial^2 v}{\partial t^2} = 0 \qquad (1.2)$$

are to be solved. Here u, v are the displacement components along the fixed axes x and y; t is the time; λ, μ are Lame's coefficients; ρ is the density of the wedged material, $\theta = \partial u/\partial x + \partial u/\partial y$. Definite boundary conditions are assumed on the surface of the crack. Due to the thinness of the crack the boundary conditions may be referred to the cut $ABOB'A'$; for the general nonstationary wedging problem these conditions, without taking account of molecular cohesion forces, are of the form

$$
\begin{aligned}
v = F(x, t), \qquad \tau_{xy} - k\sigma_y = 0 \qquad &\text{on } AB \\
\tau_{xy} = 0, \qquad \sigma_y = 0 \qquad &\text{on } BO \text{ and } OB' \\
v = -F(x, t), \qquad \tau_{xy} - k\sigma_y = 0 \qquad &\text{on } B'A'
\end{aligned}
\tag{1.3}
$$

where $F(x, t)$ is a function determining the equation of the moving surface of the crack in a fixed coordinate system related to the wedged body; σ_x, σ_y, τ_{xy} are the components of the stress tensor. For the problem considered it is natural to pass to a moving coordinate system $\xi\eta$, related to the moving wedge

$$
\xi = x + Vt, \qquad \eta = y \tag{1.4}
$$

and the origin of the system of coordinates $\xi\eta$ is conveniently taken (Fig. 1) at the end of the crack O. Let us denote by l_1 the distance from the front end of the wedge C to the end of the crack O_1 and by l_2 the distance from the meeting points B and B' to the end of the crack. The boundary conditions may then be written in the form

$$
\begin{aligned}
\tau_{\xi\eta} = 0, \ \sigma_\eta = 0 \qquad &(0 \leqslant \xi \leqslant l_2, \ \eta = 0) \\
v = \pm f(\xi - l_1), \qquad \tau_{\xi\eta} - k\sigma_\eta = 0 \qquad &(l_2 < \xi < \infty, \ \eta = 0)
\end{aligned}
\tag{1.5}
$$

where $f(t)$ is a function which determines the equation of the surface of the wedge in the moving coordinate system with the origin at the front point of the wedge C, i.e. the function which determines the shape of the wedge; the plus and minus signs correspond to the upper and lower surface of the crack, respectively. The location of the meeting points B and B' is determined in the case of a wedge with a rounded front part from the condition of finiteness of the stress σ_η at these points, by analogy to the well-known Muskhelishvili condition in the problem on stamp indentation. The location of the end of the crack O with respect to the front point of the wedge C is determined from the Khristianovich condition on finiteness of stresses and smooth closing of opposite sides of the crack at the point O. Just as in static problems [1], it becomes necessary that the fracture stress σ_η, calculated without taking molecular cohesion forces into account which are acting in the vicinity of O, (i.e. on the basis of the boundary-value problem (1.1), (1.2), (1.5)), approaches infinity in accordance with the law $K/\pi\sqrt{s}$, where s is the distance to this point and K is the cohesion modulus of the wedged material.

As is seen, the problem investigated represents a peculiar combination of the problem of the indentation of a uniformly moving stamp and of the crack problem. We note that the problem of stamps, moving on the boundary of a semispace, was considered in papers by Galin [2,3] and Radok [4]. Ioffe [5] and Radok [4] considered also the physically unrealistic problem of a uniformly moving crack of finite length, with constant stresses at infinity. The length of the crack was assumed to be given; the question of determining this length was not touched upon by these authors.

To us it seems that the problem considered is basic for the theory of cutting. In all models of the process of cutting known to the authors it is assumed that the cutter is in complete contact with the body cut. Such a model is valid, however, only for low cutting speeds; as the cutting speed is increased the fracture mechanism becomes brittle or quasi-brittle* and the cutting process is necessarily accompanied by wedging of the material. The model assumed here is confirmed by the fact observed in practice, namely that the cutters are worn mostly along the sides**.

2. Solution of the general problem. In view of the obvious symmetry of the problem with respect to the ξ-axis it is sufficient to consider only the lower half-plane, taking the conditions on its boundary in the form

$$\begin{aligned} v &= 0, \qquad \tau_{\xi\eta} = 0 \qquad (-\infty < \xi \leqslant 0) \\ \tau_{\xi\eta} &= 0, \qquad \sigma_\eta = 0 \qquad (0 \leqslant \xi < l_2) \\ v &= -f(\xi - l_1), \qquad \tau_{\xi\eta} - k\sigma_\eta = 0 \qquad (l_2 \leqslant \xi < \infty) \end{aligned} \tag{2.1}$$

We recall briefly the basic relation of the method of Galin [3] which will be used in the sequel. We note that in Section 9 of [3] which is of interest to us, inaccuracies are contained which have influenced the final formulas; these inaccuracies are rectified in the present paper. The stresses and displacements are expressed in the following form:

$$\begin{aligned} &u = -2A\,\mathrm{Im}\,\varphi(z_1) - 2B\,\mathrm{Im}\,\psi(z_2), \qquad v = 2C\,\mathrm{Re}\,\varphi(z_1) + 2D\,\mathrm{Re}\,\psi(z_2) \\ &\sigma_\xi = -2L\,\mathrm{Im}\,\varphi'(z_1) - 2F\,\mathrm{Im}\,\psi'(z_2), \qquad \sigma_\eta = -2G\,\mathrm{Im}\,\varphi'(z_1) - 2H\,\mathrm{Im}\,\psi'(z_2) \\ &\tau_{\xi\eta} = 2M\,Re\,\varphi'(z_1) + 2N\,\mathrm{Re}\,\psi'(z_2) \end{aligned} \tag{2.2}$$

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* Plastic deformations occur in quasi-brittle fracture, but they are limited to a thin layer near the crack surface.

** S.A. Khristianovich repeatedly called attention to the necessity of taking wedging into account in problems of cutting.

where

$$A = -\frac{1}{1-2\nu}\sqrt{1-\frac{1-2\nu}{2(1-\nu)}m^2}, \qquad B = -\frac{1}{1-2\nu}\sqrt{1-m^2} \tag{2.3}$$

$$C = \frac{1}{1-2\nu}\left[1-\frac{1-2\nu}{2(1-\nu)}m^2\right], \qquad D = \frac{1}{1-2\nu}$$

$$L = -\frac{E}{(1+\nu)(1-2\nu)}\left(1+\frac{\nu m^2}{2(1-\nu)}\right)\sqrt{1-\frac{1-2\nu}{2(1-\nu)}m^2}$$

$$F = -\frac{E}{(1+\nu)(1-2\nu)}\sqrt{1-m^2}, \qquad G = \frac{E(2-m^2)}{2(1+\nu)(1-2\nu)}\sqrt{1-\frac{1-2\nu}{2(1-\nu)}m^2}$$

$$H = \frac{E}{(1+\nu)(1-2\nu)}\sqrt{1-m^2}$$

$$M = \frac{E}{(1+\nu)(1-2\nu)}\left[1-\frac{1-2\nu}{2(1-\nu)}m^2\right], \qquad N = \frac{E}{(1+\nu)(1-2\nu)}\left(1-\frac{m^2}{2}\right)$$

$$k_1^2 = 1-\frac{V^2}{c_1^2}, \qquad k_2^2 = 1-\frac{V^2}{c_2^2}, \qquad c_1^2 = \frac{\lambda+2\mu}{\rho}, \qquad c_2^2 = \frac{\mu}{\rho}, \qquad m = \frac{V}{c_2}$$

(E is Young's modulus, ν is Poisson's ratio of the wedged material) and c_1 and c_2 are the velocities of the dilatational and equivoluminal waves, respectively. The functions $\phi(z_1)$ and $\psi(z_2)$ represent analytic functions of complex variables

$$z_1 = \xi + ik_1\eta, \qquad z_2 = \xi + ik_2\eta \tag{2.4}$$

related to other analytic functions w_1 and w_2 by linear relations

$$\varphi'(z_1) = \frac{N}{2\pi(GN-MH)}w_1(z_1) - \frac{iH}{2\pi(GN-MH)}w_2(z_1)$$
$$\psi'(z_2) = -\frac{M}{2\pi(GN-MH)}w_1(z_2) + \frac{iG}{2\pi(GN-MH)}w_2(z_2) \tag{2.5}$$

and the functions $w_1(z)$ and $w_2(z)$ in turn are determined by the formulas

$$w_1(z) = u_1 - iv_1 = \int_{-\infty}^{\infty}(\sigma_\eta)_{\eta=0}\frac{d\zeta}{\zeta - z}, \qquad w_2(z) = u_2 - iv_2 = \int_{-\infty}^{\infty}(\tau_{\xi\eta})_{\eta=0}\frac{d\zeta}{\zeta - z} \tag{2.6}$$

Further, the relation is valid

$$\left(\frac{\partial v}{\partial \xi}\right)_{\eta=0} = \frac{CN-DM}{GN-MH}\frac{1}{\pi}\int_{-\infty}^{\infty}(\sigma_\eta)_{\eta=0}\frac{d\zeta}{\zeta-\xi} + \frac{CH-DG}{GN-MH}(\tau_{\xi\eta})_{\eta=0} \tag{2.7}$$

On the boundary of the half-plane, at $z = \xi$ we have obviously

$$u_1 = \int_{-\infty}^{\infty}(\sigma_\eta)_{\eta=0}\frac{d\zeta}{\zeta-\xi}, \qquad v_1 = \pi(\sigma_\eta)_{\eta=0} \tag{2.8}$$

Using relations (2.7) and (2.8), we reduce the boundary-value problem (2.1) to the following Hilbert problem for the function

$$u_1 = 0 \qquad (-\infty < \xi \leqslant 0), \qquad v_1 = 0 \quad (0 < \xi < l_2)$$
$$-p\pi f'(\xi - l_1) = u_1 + kqv_1 \qquad (l_2 \leqslant \xi < \infty) \tag{2.9}$$

where the notation is introduced

$$p = \frac{GN - MH}{CN - DM} = \frac{E}{1+\nu}\frac{2}{m^2}\left\{\sqrt{1-m^2}\sqrt{1 - \frac{1-2\nu}{2(1-\nu)}m^2} - \right.$$
$$\left. - \left(1 - \frac{m^2}{2}\right)^2\right\}\left[1 - \frac{1-2\nu}{2(1-\nu)}m^2\right]^{-1/2} \tag{2.10}$$
$$q = \frac{CH - DG}{CN - DM} = \frac{2}{m^2}\left\{\left(1 - \frac{m^2}{2}\right) - \sqrt{1-m^2}\sqrt{1 - \frac{1-2\nu}{2(1-\nu)}m^2}\right\}\left[1 - \frac{1-2\nu}{2(1-\nu)}m^2\right]^{-1/2}$$

The general methods of solving Hilbert's problem are considered in detail in the monographs by Muskhelishvili [6], Galin [3] and Gakhov [7]; we will therefore not dwell on the solution of problem (2.9) but present the ready answer. We note that the solution sought must satisfy the physically natural integration condition at the points of discontinuity of the coefficients in Hilbert's problem, as well as the condition of approaching zero at infinity. These conditions determine the unique solution of Hilbert's problem which is of the form

$$w_1(z) = \frac{c_0 - \Phi(z)}{z^{1/2}(l_2 - z)^{1-\theta}}, \qquad \Phi(z) = p \sin \pi\theta \int_{l_2}^{\infty} \frac{f'(t - l_1)(t - l_2)^{1-\theta} t^{1/2} dt}{t - z} \tag{2.11}$$

where c_0 is a real positive constant, and

$$\theta = \frac{1}{\pi} \tan^{-1} \frac{1}{kq} \tag{2.12}$$

(the value of $\tan^{-1}$ is taken smaller than $\pi/2$). In particular, in the absence of friction at the sides of the wedge, i.e. for $k = 0$, Hilbert's boundary-value problem degenerates into a mixed problem, $\theta = 1/2$ and the solution (2.11) takes on the form

$$w_1(z) = \frac{c_0 - \Phi(z)}{\sqrt{z(l_2 - z)}}, \qquad \Phi(z) = p \int_{l_2}^{\infty} \frac{f'(t - l_1)\sqrt{t(t - l_2)}\, dt}{t - z} \tag{2.13}$$

From solution (2.11) and the second relation (2.8), we obtain the general expression for the normal stress at the boundary

$$(\sigma_\eta)_{\eta=0} = \begin{cases} \dfrac{c_0 - \Phi(\xi)}{\pi(-\xi)^{1/2}(l_2 - \xi)^{1-\theta}} & (-\infty < \xi \leqslant 0) \\ 0 & (0 < \xi < l_2) \\ -\dfrac{\sin \pi\theta\, [c_0 - \Phi(\xi)]}{\pi(\xi - l_2)^{1-\theta}\xi^{1/2}} - \dfrac{1}{2} p \sin 2\pi\theta f'(\xi - l_1) & (l_2 \leqslant \xi < \infty) \end{cases} \tag{2.14}$$

where

$$\Phi(\xi) = p \sin \pi\theta \int_{l_2}^{\infty} \frac{f'(t-l_1)(t-l_2)^{1-\theta}\, t^{1/2} dt}{t-\xi} \qquad (2.15)$$

(the singular integral is taken in the sense of its principal value). In particular, in the absence of friction, Expressions (2.14) and (2.15) take on the form

$$(\sigma_\eta)_{\eta=0} = \begin{cases} \dfrac{c_0 - \Phi(\xi)}{\pi\sqrt{(l_2-\xi)(-\xi)}} & (-\infty < \xi \leqslant 0) \\ 0 & (0 < \xi < l_2) \\ -\dfrac{c_0 - \Phi(\xi)}{\pi\sqrt{\xi(\xi-l_2)}} & (l_2 \leqslant \xi < \infty) \end{cases} \qquad (2.16)$$

$$\Phi(\xi) = p \int_{l_2}^{\infty} \frac{f'(t-l_1)\sqrt{t(t-l_2)}\, dt}{t-\xi} \qquad (2.17)$$

Further, integrating (2.7), we obtain an expression for the displacement v of the boundary of the half-plane $\xi = s$:

$$v(s) + c = \frac{1}{p\pi} \int_{-\infty}^{\infty} (\sigma_\eta)_{\eta=0} \ln \left| \frac{\zeta - s_0}{\zeta - s} \right| d\zeta + \frac{q}{p} \int_{s_0}^{s} (\tau_{\xi\eta})_{\eta=0} d\xi \qquad (2.18)$$

where c, s_0 are constants of integration. We note that negative s correspond to the part of the body, not as yet cut, and the displacement v for such s is equal to zero. Therefore, on the strength of (2.18), for $s > 0$ the relation is valid

$$c = \frac{1}{p\pi} \int_{-\infty}^{\infty} (\sigma_\eta)_{\eta=0} \ln \left| \frac{\zeta - s_0}{\zeta + s} \right| d\zeta + \frac{q}{p} \int_{s_0}^{-s} (\tau_{\xi\eta})_{\eta=0} d\xi \qquad (2.19)$$

Subtracting (2.19) from (2.18) and using the boundary conditions, we obtain the final expression for the displacement v in the form

$$v = \frac{1}{p\pi} \int_{-\infty}^{\infty} (\sigma_\eta)_{\eta=0} \ln \left| \frac{\zeta + s}{\zeta - s} \right| d\zeta + \frac{qk}{p} \int_{l_2}^{s} (\sigma_\eta)_{\eta=0} d\xi \qquad (2.20)$$

In the absence of friction, the expression for the displacement v takes on the form

$$v = \frac{1}{p\pi} \int_{-\infty}^{\infty} (\sigma_\eta)_{\eta=0} \ln \left| \frac{\zeta + s}{\zeta - s} \right| d\zeta \qquad (2.21)$$

3. Determination of constants entering the solution. General dynamic condition at the end of the crack. 1. The solution obtained contains three undetermined real constants c_0, l_1 and l_2. For their determination we use three conditions not employed as yet:

1 – The thickness of the wedge is equal to $2h$ at infinity.

2 – The stress at the meeting point of the crack surface and the wedge is finite for a wedge with a rounded front part.

3 – The stress at the end of the crack is finite, or, which is the same, the opposite sides of the crack close smoothly at its end.

We require that the derivative $f'(\xi)$ approaches zero at infinity faster than $\xi^{-3/2+\theta}$. Then the integral (2.15) is known to exist and approaches zero as $\xi \to \infty$. In the analysis of the first condition we consider separately the cases of absence of friction ($k = 0$) and presence of friction ($k \pm 0$). In the absence of friction, the expression for normal stresses at the boundary is of the form (2.16), and the displacement v is represented by Formula (2.21). By virtue of the first condition we have

$$h = \frac{I}{p\pi}, \qquad I = \lim_{s\to\infty}\left\{\int_{-\infty}^{\infty}(\sigma_\eta)_{\eta=0}\ln\left|\frac{\zeta - s}{\zeta + s}\right| d\zeta\right\} \tag{3.1}$$

As was done in [8], it can be shown that

(a) If the function $(\sigma_\eta)_{\eta=0}$ is finite, i.e. becomes zero for all those ζ for which $|\zeta|$ is larger than a certain A, then $I = 0$.

(b) If the function $(\sigma_\eta)_{\eta=0}$ approaches zero as $|\zeta| \to \infty$ faster than $1/|\zeta|$, then $I = 0$.

It follows that in calculating I, only those terms of the expansion $(\sigma_\eta)_{\eta=0}$ at infinity are essential which approach zero at infinity not faster than $1/\zeta$. By virtue of the condition imposed on the function $f'(\xi)$, $\Phi(\zeta) \to 0$ as $|\zeta| \to \infty$, and we obtain from (2.16) an asymptotic formula for $(\sigma_\eta)_{\eta=0}$ as $|\zeta| \to \infty$

$$(\sigma_\eta)_{\eta=0} = -\frac{c_0}{\pi\zeta} + o\left(\frac{1}{\zeta}\right) \tag{3.2}$$

Substituting (3.2) into (3.1) and evaluating the integral [9], we find

$$h = -\frac{c_0}{\pi^2 p}\lim_{s\to\infty}\left\{\int_{-\infty}^{0} + \int_{l_2}^{\infty}\right\} = \frac{c_0}{\pi^2 p}\lim_{s\to\infty}\left\{\int_0^{\infty}\frac{1}{\sigma}\ln\frac{\sigma+1}{\sigma-1}d\sigma + \int_{l_2/s}^{\infty}\frac{1}{\sigma}\ln\frac{\sigma+1}{\sigma-1}d\sigma\right\} = \frac{c_0}{p}$$

From this we obtain

$$c_0 = ph \tag{3.3}$$

In the presence of friction, the expression for normal stresses at the boundary is of the form (2.14), and the displacement v is expressed by Formula (2.20). By virtue of the first condition we have

$$h = \frac{I}{p\pi} - \frac{qk}{p}\int_{l_2}^{\infty} (\sigma_\eta)_{\eta=0} d\xi \tag{3.4}$$

where I, as before, is determined by Formula (3.1).

From (2.14) it follows that as $|\zeta| \to \infty$

$$(\sigma_\eta)_{\eta=0} = o\,(|\zeta|^{-3/2+\theta}) \tag{3.5}$$

and, since in the presence of friction θ is always smaller than 1/2, then $(\sigma_\eta)_{\eta=0}$ decreases at infinity faster than $1/\zeta$. From this and from what was said above in considering the case $k = 0$, it follows that $I = 0$ such that Formula (3.4) is written down in the form

$$h = -\frac{qk}{p}\int_{l_2}^{\infty} (\sigma_\eta)_{\eta=0}\, d\xi \tag{3.6}$$

Substituting into (3.6) the expression for $(\sigma_\eta)_{\eta=0}$ from Formulas (2.14) and (2.15), and recalling that $gk = \cot \pi\theta$, we find

$$ph = \frac{c_0 \cos \pi\theta}{\pi}\int_{l_2}^{\infty} \frac{d\xi}{\xi^{1/2}(\xi - l_2)^{1-\theta}} + p\cos^2 \pi\theta\,[h - f\,(l_2 - l_1)] -$$
$$- \frac{p \sin 2\pi\theta}{2\pi}\int_{l_2}^{\infty} \frac{d\xi}{\xi^{1/2}(\xi - l_2)^{1-\theta}}\int_{l_2}^{\infty} \frac{f'\,(t - l_1)(t - l_2)^{1-\theta}\, t^{1/2}\, dt}{t - \xi} \tag{3.7}$$

It can be shown, using tables [9.10] that

$$\int_{l_2}^{\infty} \frac{d\xi}{\xi^{1/2}(\xi - l_2)^{1-\theta}} = \frac{\Gamma\,(1/2 - \theta)\,\Gamma\,(\theta)}{\Gamma\,(1/2)\, l_2^{1/2-\theta}}$$

$$\int_{l_2}^{\infty} \frac{d\xi}{\xi^{1/2}(\xi - l_2)^{1-\theta}}\int_{l_2}^{\infty} \frac{f'\,(t - l_1)(t - l_2)^{1-\theta}\, t^{1/2}\, dt}{t - \xi}$$
$$= \frac{1}{l_2^{1/2-\theta}}\int_{l_2}^{\infty} f'\,(t - l_1)(t - l_2)^{1-\theta}\, t^{1/2} S\left(\frac{t}{l_2},\, \theta\right) dt$$

where $S(\tau,\ \theta)$ is the principal value of the integral

$$S(\tau,\theta)=\int_1^\infty \frac{d\sigma}{\sigma^{1/2}(\sigma-1)^{1-\theta}(\tau-\sigma)}$$

From this and from (3.7) we obtain the first relation for the determination of the parameters c_0, l_1 and l_2

$$c_0=\frac{\pi ph\,\Gamma(1/2)l_2^{1/2-\theta}}{\Gamma(\theta)\,\Gamma(1/2-\theta)\cos\pi\theta}-\frac{\pi p\cos\pi\theta\,[h-f(l_2-l_1)]\,\Gamma(1/2)}{\Gamma(1/2-\theta)\,\Gamma(\theta)}+ \tag{3.8}$$
$$+\frac{p\sin\pi\theta\,\Gamma(1/2)}{\Gamma(1/2-\theta)\,\Gamma(\theta)\,l_2^{1/2-\theta}}\int_{l_2}^\infty f'(t-l_1)(t-l_2)^{1-\theta}t^{1/2}S\left(\frac{t}{l_2},\theta\right)dt$$

The second relation is obtained from the condition of finiteness of normal stresses at the meeting point $\xi = l_2$ of the crack surface and the wedge. For this, as **Formula** (2.14) shows, it is necessary and sufficient that the equation

$$c_0-\Phi(l_2)=0$$

be satisfied. Thus, on the strength of (2.15), the second relation takes on the form

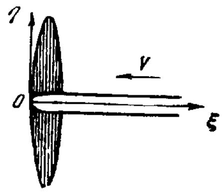

Fig. 2.

$$c_0-p\sin\pi\theta\int_{l_2}^\infty\frac{f'(t-l_1)\,t^{1/2}dt}{(t-l_2)^\theta}=0 \tag{3.9}$$

In particular, in the absence of friction ($k = 0$) it is of the form

$$h=\int_{l_2}^\infty f'(t-l_1)\sqrt{\frac{t}{t-l_2}}\,dt \tag{3.10}$$

(2) The third relation between the parameters c_0, l_1 and l_2 is obtained from the condition of finiteness of stresses and smooth closing at the end of the crack 0. It was shown for static problems [1] that to satisfy this condition, it is necessary and sufficient that the fracture stresses, calculated without taking the forces of molecular cohesion into account, be infinite at the end of the crack in accordance with the law $K/\pi\sqrt{s}$ where s is the distance from the end of the crack and K is the cohesion modulus of the material. It turns out that this last condition is valid also for dynamic problems.

To supply a proof, we consider the vicinity at the end of the propagating crack which is small as compared to the dimensions of the crack as a whole, but which is large as compared to the dimension of its end region in which the cohesion forces are acting. Because of the smallness of the dimension d as compared to the dimensions of the crack as a whole (the first hypothesis [1]), such a region may always be found. Further, for a small interval of time it may always be assumed that the end of the crack is propagating with constant velocity. Thus, to determine the influence of the forces of molecular cohesion on stresses and deformations it is sufficient to consider a semi-infinite cut moving with constant velocity V (Fig. 2) under the action of cohesion forces alone. The problem is symmetric with respect to the ξ-axis; considering again the lower-half-plane, we obtain the boundary conditions on the ξ-axis in the form*

$$\begin{gathered} v^{(a)} = 0, \qquad \tau_{\xi\eta}^{(a)} = 0 \qquad (-\infty < \xi \leqslant 0) \\ \sigma_\eta^{(a)} = G(\xi), \qquad \tau_{\xi\eta}^{(a)} = 0 \qquad (0 \leqslant \xi \leqslant d) \\ \sigma_\eta^{(a)} = \tau_{\xi\eta}^{(a)} = 0 \qquad (\xi > d) \end{gathered} \tag{3.11}$$

where $G(\xi)$ is the distribution of the cohesion forces which are equal to zero outside the end region.

The determination of the corresponding stresses and deformations, in complete analogy to Section 2, is reduced to the solution of the mixed problems for the analytic function $w_1^{(a)}(z) = u_1^{(a)} - iv_1^{(a)}$

$$u_1^{(a)} = 0 \; (-\infty < \xi \leqslant 0), \quad v_1^{(a)} = \pi G(\xi) \quad (0 \leqslant \xi \leqslant d), \quad v_1^{(a)} = 0 \quad (\xi > d) \tag{3.12}$$

In accordance with the formula of Keldysh-Sedov [11], the function $w_1^{(a)}(z)$ is represented in the form

$$w_1^{(a)}(z) = \frac{1}{\sqrt{z}} \int_0^d \frac{G(t)\, t^{1/2} dt}{t - z} \tag{3.13}$$

such that, in accordance with (2.8), the stress $(\sigma_\eta^{(a)})_{\eta=0}$ at the point $\xi = -s$ $(s > 0)$ is expressed by the formula

$$(\sigma_\eta^{(a)})_{\eta=0} = -\frac{1}{\pi\sqrt{s}} \int_0^d \frac{G(t) t^{1/2}\, dt}{t + s}$$

For small s, due to autonomy of the end region (i.e. the independence of its form and the cohesion forces acting in it of loading; the second hypothesis [1]), and by definition of the cohesion modulus [1], we have

$$\sigma_\eta^{(a)} = -\frac{1}{\pi\sqrt{s}} \int_0^d \frac{G(t) dt}{\sqrt{t}} = -\frac{K}{\pi\sqrt{s}} \tag{3.14}$$

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* The superscript (a) designates the components of stress, displacement, etc., produced by the cohesion forces.

To ensure the finiteness of the fracture stress at the end of the crack, the fracture stress σ_η, calculated without taking the forces of cohesion into account, should compensate the stress $\sigma_\eta^{(a)}$, i.e. should approach infinity in accordance with the law

$$\sigma_\eta = \frac{K}{\pi \sqrt{s}} \tag{3.15}$$

The study shows that the fulfilment of condition (3.15) simultaneously ensures the finiteness of the stress σ_ξ and the smoothness of closing of opposite sides at the end of the crack. In order that the solution of the problem considered satisfies condition (3.15) it is necessary, as Formula (2.14) indicates, to satisfy the relation

$$c_0 - \Phi(0) = K l_2^{1-\theta}$$

or, in view of (2.15), the relation

$$c_0 - p \sin \pi\theta \int_{l_2}^{\infty} \frac{f'(t-l_1)(t-l_2)^{1-\theta} dt}{t^{1/2}} = K l_2^{1-\theta} \tag{3.16}$$

which is indeed the third relation connecting the parameters c_0, l_1 and l_2. In the absence of friction, condition (3.16) takes on the form

$$h - \int_{l_2}^{\infty} f'(t-l_1) \sqrt{\frac{t-l_2}{t}}\, dt = \frac{K\sqrt{l_2}}{p} \tag{3.17}$$

Subtracting (3.9) from (3.16), we obtain the equation

$$\int_{l_2}^{\infty} \frac{f'(t-l_1)\, dt}{t^{1/2}(t-l_2)^{\theta}} = \frac{K}{l_2^{\theta}\, p \sin \pi\theta} \tag{3.18}$$

In particular, in the absence of friction, this equation takes on the form

$$\int_{l_2}^{\infty} \frac{f'(t-l_1)\, dt}{\sqrt{t(t-l_2)}} = \frac{K}{p\sqrt{l_2}} \tag{3.19}$$

Conditions (3.3), (3.1), and (3.17) in the absence of friction and conditions (3.8), (3.9) and (3.16) in the presence of friction for a given shape of the wedge, i.e. for a given function $f(t)$, represent finite relations which determine uniquely the values of the parameters c_0, l_1 and l_2. We note that in the absence of friction one of the conditions (3.10) and (3.17) may be replaced by Equation (3.19) and in the presence of friction one of the conditions (3.9) and (3.16) may be replaced by Equation (3.18).

4. Resistance to wedging. It is physically obvious that in the process of wedging the wedge must be subjected to some force which must be directed along the axis of the wedge. This force, which is equal to the resistance of the body to wedging, will be denoted by R; it is composed of the resistance of friction R_1 which is the resultant of the forces of friction applied at the side of the wedge, and of frontal resistance R_2 which is the resultant of the projections of normal forces on the wedge axis. The friction resistance R_1 may be represented in the form

$$R_1 = -2\int_{l_2}^{\infty} (\tau_{\xi\eta})_{\eta=0}\, d\xi \tag{4.1}$$

Formula (3.6) may be written in the following manner:

$$h = -\frac{q}{p}\int_{l_2}^{\infty} (\tau_{\xi\eta})_{\eta=0}\, d\xi \tag{4.2}$$

Comparing (4.2) and (4.1) we obtain a very simple formula for the friction resistance:

$$\begin{aligned} R_1 &= \frac{2hp}{q} = 2hp_0\left(\frac{p}{p_0 q}\right) \\ &= 2hp_0 \frac{2(1-\nu)\left[\left(1-\frac{1}{2}m^2\right)^2 - \sqrt{1-m^2}\sqrt{1-\frac{1-2\nu}{2(1-\nu)}m^2}\right]}{\left(1-\frac{1}{2}m^2\right) - \sqrt{1-m^2}\sqrt{1-\frac{1-2\nu}{2(1-\nu)}m^2}} \\ &\left(p_0 = \frac{E}{2(1-\nu^2)} = (p)_{m=0}\right) \end{aligned} \tag{4.3}$$

The dependence of $R_1^* = R_1/2hp_0$ on m for different ν is represented in Fig. 3; as is seen, the resistance to friction decreases with an

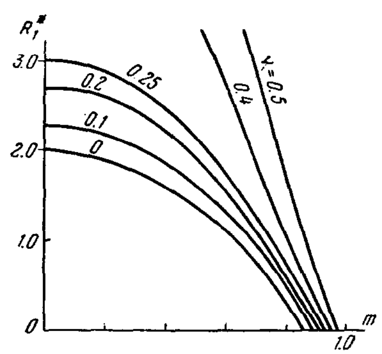

Fig. 3.

increase in the velocity of motion of the wedge tending to zero as the critical velocity is approached which corresponds to vanishing of p (or coinciding with the velocity of propagation of Rayleigh surface waves, see below). It is remarkable that the expression for the force of friction obtained does not depend on the magnitude of the coefficient of friction or the shape of the wedge but is completely determined by the thickness of the wedge at infinity, the velocity of wedging and the elastic characteristics of the wedged material.

Since the projection of the normal stress on the wedge axis is equal to $-(\sigma_\eta)_{\eta=0} f'(\xi)$ in view of the thinness of the wedge, the frontal resistance R_2 is determined by the relation

$$R_2 = -2\int_{l_2}^{\infty} (\sigma_\eta)_{\eta=0} f'(\xi)\,d\xi \tag{4.4}$$

5. Solution of specific problems. Let us consider several particular problems which have an interest of their own.

(1) *Wedge of constant thickness* (Fig. 4). In this case $f'(t) \equiv 0$ and the function $w_1(z)$ is written in the form

$$w_1(z) = \frac{c_0}{z^{1/2}(l_2 - z)^{1-\theta}} \tag{5.1}$$

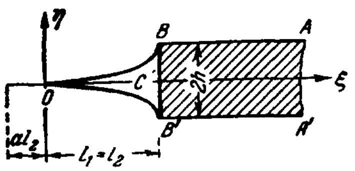

Fig. 4.

where the real positive constant c_0 is determined by the relation

$$c_0 = \frac{\pi p h l_2^{1/2-\theta}\Gamma(1/2)}{\Gamma(\theta)\Gamma(1/2-\theta)\cos\pi\theta}, \qquad \theta = \frac{1}{\pi}\tan^{-1}\frac{1}{kq} \tag{5.2}$$

To determine the constants $l_1 = l_2$ we use relation (3.16) which in our case takes on the form

$$c_0 = K l_2^{1-\theta} \tag{5.3}$$

Using relation (5.2), we obtain an expression for $l_2 = l_1$ in the form

$$l_1 = l_2 = \frac{\pi^2 p^2 h^2 \Gamma^2(1/2)}{\Gamma^2(\theta)\Gamma^2(1/2-\theta)\cos^2\pi\theta K^2} \tag{5.4}$$

The distribution of normal stresses on the sides of the wedge and on the crack extension is of the form

$$(\sigma_\eta)_{\eta=0} = \begin{cases} \dfrac{c_0}{\pi(-\xi)^{1/2}(l_2-\xi)^{1-\theta}} & (-\infty<\xi\leqslant 0) \\ 0 & (0<\xi<l_2) \\ -\dfrac{c_0}{\pi\xi^{1/2}(\xi-l_2)^{1-\theta}} & (l_2\leqslant\xi<\infty) \end{cases} \tag{5.5}$$

In the absence of friction, the function $w_1(z) = w_1^*(z)$ is written in the form

$$w_1^*(z) = \frac{c_0^*}{\sqrt{z(l_2^*-z)}}, \qquad c_0^* = ph \tag{5.6}$$

and the free length of the crack in the absence of friction l_2^* is expressed by the formula

$$l_2^* = \frac{p^2h^2}{K^2} \tag{5.7}$$

It is convenient to represent the last expression in the form

$$\frac{l_2^*}{l_{20}^*} = \frac{p^2}{p_0^2}, \qquad p_0 = (p)_{m=0} = \frac{E}{2(1-\nu^2)} \tag{5.8}$$

where l_{20}^* is the free length of the crack in the case of a wedge of constant thickness at rest, which, as shown in [1], is determined by the relation

$$l_{20}^* = \frac{E^2h^2}{4(1-\nu^2)^2K^2}$$

A graph of the function l_2^*/l_{20}^* in dependence on $m = V/c_2$ for a value of Poisson's ratio $\nu = 0.25$ is given in Fig. 5. We see that as the critical velocity is reached which corresponds to a vanishing of p, the length of the free part of the crack also vanishes. At a speed which exceeds the critical one, our formulation of the problem, as will be shown below, becomes invalid.

Relationship (5.4) may be represented in view of (5.7) in the form

$$\frac{l_2}{l_2^*} = \frac{\pi^3}{\Gamma^2(\theta)\Gamma^2(1/2-\theta)\cos^2\pi\theta} \tag{5.9}$$

A graph of the dependence of l_2/l_2^* on the parameter θ which characterizes friction is given in Fig. 6.

(2) *Wedge with a rounded front part.* To estimate the influence of a rounding of the front part of the wedge, we consider a wedge whose shape is given by the relation

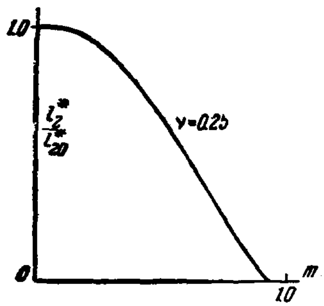

Fig. 5.

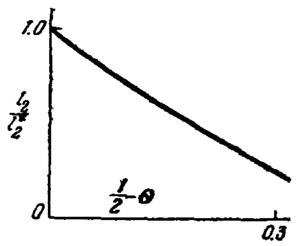

Fig. 6.

$$f(\xi - l_1) = \begin{cases} h & (\xi - l_1 > B) \\ h\left[1 - \dfrac{(B - \xi + l_1)^2}{B^2}\right] & (0 < \xi - l_1 < B) \end{cases} \tag{5.10}$$

where B is the length of the part of the wedge which is rounded off. Friction is neglected. It is obvious that with $B = 0$ we obtain a wedge of constant thickness. Without writing the function $w_1(z)$ explicitly, we present only the equations which determine the unknown parameters l_1 and l_2. Using relations (3.10) and (3.17), we find

$$\frac{2B^2}{\beta l_2} = -l_2\left[1 - \sqrt{\beta^2 - 1}\sqrt{4\beta^2(\beta^2 - 1) + 1}\right] + 2(B + l_1 - l_2)\left(1 + \sqrt{\beta^2 - 1}\right) \tag{5.11}$$

$$1 - \frac{K\sqrt{l_2}}{ph} = \frac{2\beta}{B^2}\left[(B + l_1)l_2\left(1 - \sqrt{\beta^2 - 1}\right) - \frac{l_2^2}{4}\left(1 - \sqrt{\beta^2 - 1}\sqrt{4\beta^2(\beta^2 - 1) + 1}\right)\right]$$

where

$$\beta = \operatorname{arch}\sqrt{\frac{l_1 + B}{l_1}}$$

To estimate the influence of the rounding off, it is sufficient to consider the case of small B. In this case the second equation (5.11) yields

$$l_2 = l_1 + B - \tfrac{1}{2}B\sqrt[3]{\frac{B}{l_1}} \tag{5.12}$$

Substituting (5.12) into the first equation (5.11) and discarding terms of order of magnitude larger than the first, we find

$$l_1 = \frac{p^2h^2}{K^2} + O\left(B\sqrt[3]{\frac{B}{l_1}}\right) = l_1^* + O\left(B\sqrt[3]{\frac{B}{l_1^*}}\right) \qquad (5.13)$$

where $l_1^* = l_2^*$ is the length of the free part of the crack for a wedge of constant thickness. We see that a small rounding of the edges of the wedge is of small influence on the length of the free part of the crack.

(3) *Wedge rounded off in accordance with a power law.* Consider a wedge whose equation is given in the form

$$f(\xi - l_1) = h\left[1 - \frac{A^n}{(A - l_1 + \xi)^n}\right] \qquad (\xi - l_1 \geqslant 0) \qquad (5.14)$$

where A is a positive constant. We again neglect friction on the sides of the wedge. Then relations (3.10) and (3.17) which determine l_1 and l_2 take on the form

$$l_2^n = nA^nF_1(\beta), \qquad \frac{K\sqrt{l_2}}{ph} = \frac{nA^n}{l_2^n}F_2(\beta), \qquad \beta = \frac{l_1 - A}{l_2} < 1$$

$$F_1(\beta) = \int_1^\infty \frac{\xi^{1/2}d\xi}{\sqrt{\xi - 1}\,(\xi - \beta)^{n+1}} = \frac{B\left(\frac{1}{2}, n\right)F\left(n+1, \frac{1}{2}, n+\frac{1}{2}, -\frac{\beta}{1-\beta}\right)}{(1-\beta)^{n+1}} \qquad (5.15)$$

$$F_2(\beta) = \int_1^\infty \frac{d\xi}{\sqrt{\xi(\xi - 1)}\,(\xi - \beta)^{n+1}} = \frac{B\left(\frac{1}{2}, n+1\right)F\left(n+1, \frac{1}{2}, n+\frac{3}{2}, -\frac{\beta}{1-\beta}\right)}{(1-\beta)^{n+1}}$$

where $F(a_1, a_2, a_3, z)$ is a hypergeometric function and $B(a_1, a_2)$ is Euler's beta function; the values of integrals are taken from [10]. Eliminating l_2 from Equations (5.15), we obtain an equation for β in the form

$$\frac{K\sqrt{A}}{ph} = F(\beta) = F_2(\beta)\,n^{-\frac{1}{2n}}\,[F_1(\beta)]^{-\frac{2n+1}{2n}} \qquad (5.16)$$

Having determined β from this equation, we find l_2 from the first equation (5.15). Since this example is of considerable interest, its more detailed analysis will be presented in a separate paper.

6. Limiting velocity of crack propagation. In the problem of wedging considered, as well as the problems of a stamp [die] which is moving with constant velocity on the surface of a semi-space considered in the works of Galin [2,3], the dependence of the solution on the velocity V is determined chiefly by a dependence of the dimensionless velocity of the motion of a wedge or a die $m = V/c_2$ on the constants p and q determined by relation (2.10).

As the present study indicates, in the range $0 < \nu < 1/2$, $0 < m \leqslant 1$ the quantity q is finite and positive, while the quantity p vanishes as the critical velocity is reached which is determined by equation (Fig. 7)

$$\sqrt{1-m_0^2}\sqrt{1-\frac{1-2\nu}{2(1-\nu)}m_0^2}-\left(1-\frac{1}{2}m_0^2\right)^2=0 \tag{6.1}$$

($m_0 = V_0/c_2$, V_0 is the critical velocity), and becomes negative beyond it. It is remarkable that Equation (6.1) coincides with the equation which determines the velocity of Rayleigh surface waves in the given material (see, for example, [12]). This fact is of principal significance in the problems considered, because it limits the applicability of the formulation of the problems studied to velocities smaller than the Rayleigh velocity. Let us show this with the example of a moving wedge of constant thickness (Section 5) and a moving stamp with a plane base in the absence of friction.

We obtain the following expressions for the stresses σ_ξ and σ_η at the point $\xi = -al_2$ located at a distance $l_2(1+a)$ from the front part of the wedge (Fig. 4), if we consider the wedging by a wedge of constant thickness $2h$:

$$\sigma_\xi = \frac{\nu m^2K^2E}{2p^2h\ (1-\nu^2)\sqrt{a(1+a)\left(1-\frac{1-2\nu}{2(1-\nu)}m^2\right)}}, \qquad \sigma_\eta = \frac{K^2}{ph\sqrt{a(1+a)}} \tag{6.2}$$

As the Rayleigh velocity is approached, these stresses tend to infinity since $p \to 0$, whereby the tensile stress σ_ξ approaches infinity faster than the tensile stress σ_η. Inasmuch as no material can withstand infinite tensile stresses, this indicates that in front of the running crack transverse cracks will be formed whose appearance will completely change the pattern of motion because the model of stationary motion assumed by us will no longer correspond to reality. Phenomena which occur as the velocity of motion of the wedge is close to the Rayleigh velocity represent in their nature a peculiar resonance. We note now that from Formula (3.8) it follows that for any wedge as $p \to 0$, the constant c_0 approaches zero. From (3.16) it follows that as $p \to 0$ the constant l_2 approaches zero. Thus the length of the free part of the crack approaches zero as the Rayleigh velocity is reached, and we formulate the important conclusion: the velocity of propagation of cracks in a given material may not exceed the velocity of propagation of Rayleigh surface waves in this material.

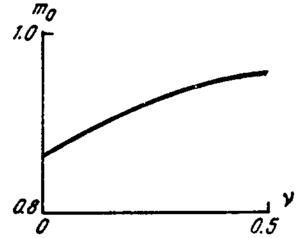

Fig. 7.

For velocities of the wedge which exceed the Rayleigh velocity, the formulation of the wedging problem changes essentially: the wedge must necessarily be assumed in complete contact with the wedged body; no free parts of the surface are formed in this case. It seems to us also that a study of wedging problems, motions of dies [stamps] and other similar problems should be based for near-Rayleigh velocities on an essentially new model of the wedging process of the body.

In addition to the Rayleigh velocity, there exists another lower critical velocity for isotropic bodies. It is natural to assume that in an isotropic body the free development of a crack proceeds in the direction of maximum cleavage stress. Therefore, in order that in the wedging problem the crack be developed in front of the wedge and that it remain rectilinear, it is necessary that the extension of the cut (negative part of the x-axis) be the line of maximum cleavage stresses, at least in the neighborhood of the end of the crack. Let us now draw a circle of some radius r which is small as compared to the dimensions of the free part of the crack, but which is large as compared to the dimensions of the end region, with its center at the end of the crack. Let γ be the angle measured from the extension of the cut such that for small γ the cleavage stress may be represented in the form

$$\sigma_{\gamma\gamma} = \frac{K}{\pi \sqrt{r}\,(GN - MH)} \{GN - MH - \alpha\gamma^2 + 0(\gamma^4)\} \tag{6.3}$$

where

$$\begin{aligned}
\alpha &= FM - LN + \tfrac{3}{4}(GN - MH) + \tfrac{3}{8}(GNk_1^2 - MHk_2^2) - MN(k_1 - k_2) \\
&= \frac{E^2 f(m, \nu)}{(1+\nu)^2 (1-2\nu)^2} \sqrt{1 - \frac{1-2\nu}{2(1-\nu)} m^2} \\
f(m, \nu) &= (1 - \tfrac{1}{2} m^2)(1 + \tfrac{\nu}{2(1-\nu)} m^2) - \sqrt{1-m^2}\sqrt{1 - \tfrac{1-2\nu}{2(1-\nu)} m^2} + \\
&\quad + \tfrac{3}{4}\left[(1 - \tfrac{1}{2} m^2)^2 - \sqrt{1-m^2}\sqrt{1 - \tfrac{1-2\nu}{2(1-\nu)} m^2}\right] + \\
&\quad + \tfrac{3}{8}\sqrt{1 - \tfrac{1-2\nu}{2(1-\nu)} m^2}\left[(1 - \tfrac{1}{2} m^2)^2 \sqrt{1 - \tfrac{1-2\nu}{2(1-\nu)} m^2} - (1-m^2)^{3/2}\right] - \\
&\quad - \sqrt{1 - \tfrac{1-2\nu}{2(1-\nu)} m^2}\left(1 - \tfrac{1}{2} m^2\right)\left[\sqrt{1 - \tfrac{1-2\nu}{2(1-\nu)} m^2} - \sqrt{1-m^2}\right]
\end{aligned}$$

The quantity α is positive for $0 < m < m^*$ and negative for $m^* < m < m_0$ where $m^*(\nu)$ is determined by the relation

$$f(m^*, \nu) = 0 \tag{6.4}$$

and m_0 by the relation (6.1).

We give the values of the quantity m^* for some values of ν:

$\nu =$	0	0.1	0.2	0.3	0.3	0.5
$m^* =$	0.510	0.560	0.603	0.653	0.707	0.760

Thus, for $m > m^*$, the straight line $y = 0$ becomes a line of minimum cleavage stresses, while for $m < m^*$ this line is a line of maximum cleavage stresses. Therefore, in passing through the critical velocity of motion of the wedge $V^* = m^* c_2$ the crack running in front of the wedge will become curved in an isotropic body. The consideration of the wedging problem in the formulation suggested is limited if the crack is not rectilinear *a priori*, which will be ensured by velocities which are smaller than V^*. (In an isotropic body it is possible to ensure that a crack is rectilinear, for instance, by making a rectilinear cut and then bonding it together such that at the bonded part the cohesion is weaker than in the original material. The body remains isotropic while the crack will obviously be rectilinear.) A more general assertion is valid: a crack may propagate freely in an isotropic body and remain rectilinear only for velocities which are smaller than V^*. We note that Ioffe [5] has established in a somewhat different manner the existence of this critical velocity, considering the problem of a moving crack of constant length in a homogeneous stress field. However, Equation (6.4) determining the critical velocity V^* was not indicated in her paper.

In the case when the developing crack in an isotropic body remains rectilinear for some reason, the velocity of development of the free crack is limited only by the Rayleigh velocity.

Roberts and Wells [13] have studied the maximum velocity of crack propagation in a brittle body which is in a homogeneous stress field. However, their approach, based on the solution of a static problem of the theory of elasticity, may not be accepted as suitable for quantitative calculations.

The same pattern obtains also in the problem of a moving die: this problem may be studied only for velocities of the die which are smaller than the Rayleigh velocity. Indeed, in the case of a die with a plane base the expressions for the stresses are of the form

$$\sigma_\xi = \frac{1}{p\pi}\left\{\frac{MF}{CN-DM}\operatorname{Im}[w_1(z_2)] - \frac{NL}{CN-DM}\operatorname{Im}[w_1(z_1)]\right\} \qquad (6.5)$$

$$\sigma_\eta = -\frac{1}{p\pi}\left\{\frac{GN}{CN-DM}\operatorname{Im}[w_1(z_2)] - \frac{MH}{CN-DM}\operatorname{Im}[w_1(z_1)]\right\} \qquad (6.6)$$

$$w_1(z) = -\frac{iP_0}{\sqrt{l^2-z^2}}, \qquad z_1 = \xi + ik_1\eta, \qquad z_2 = \xi + ik_2\eta$$

The coefficients C, N, M, F, G, H are determined by Formulas (2.3). As the Rayleigh velocity is approached, $p \to 0$, while the expressions in parentheses remain finite such that the stresses σ_ξ and σ_η approach infinity. Again peculiar resonance phenomena occur which are related to the fracture of the material inside the body and to a radical change of the pattern of motion. Thus also in this problem the assumed stationary model of motion appears to be suitable only for velocities which are smaller than the Rayleigh velocity. We recall that Eshelby [14] established that the velocity of propagation of Rayleigh waves is the upper limit of the velocity of motion of linear dislocation in the material if the atomic nature of the material is taken into account.

The results obtained in the present paper once more confirm the significance of the Rayleigh velocity in problems of dynamics of a solid body.

BIBLIOGRAPHY

1. Barenblatt, G.I., O ravnovesnykh treshchinakh, obrazuiushchikhsia pri khrupkom razrushenii (On equilibrium cracks, formed in brittle fracture). *PMM* Vol. 23, No. 3, 4, 5, 1959.

2. Galin, L.A., Smeshannye zadachi teorii uprugosti s silami treniia dlia poluploskosti (Mixed problems of the theory of elasticity with friction forces for a half-plane). *Doklady Akad. Nauk* Vol. 39, No. 3, 1943.

3. Galin, L.A., *Kontaktnye zadachi teorii uprugosti (Contact Problems of the Theory of Elasticity)*. GITTL, Moscow, 1953.

4. Radok, J.R.M., On the solutions of problems of dynamic plane elasticity. *Quarterly of Applied Mathematics* Vol. 14, No. 3, Oct. 1956.

5. Ioffe, E., The moving Griffith crack, *Phil Mag.* Vol. 42, No. 330, 1951.

6. Muskhelishvili, N.I., *Singuliarnye integralnye uravneniia (Singular Integral Equations)*. GITTL, Moscow, 1946.

7. Gakhov, F.D., Kraevye zadachi (Boundary-value Problems). Fizmatizdat, Moscow, 1958.

8. Barenblatt, G.I. and Khristianovich, S.A., Ob obrushenii krovli pri gornykh vyrabotkakh (On the collapse of roofings of excavations). *Izvest. Akad. Nauk SSSR, Otdel. tekhn. nauk* No. 11, 1955.

9. Ryzhik, I.M. and Gradshtein, I.S., *Tablitsy integralov (Tables of Integrals)*. GITTL, Moscow-Leningrad, 1951.

10. Bateman, H., Tables of Integral Transforms. New York, McGraw-Hill, 1954.

11. Levrent'ev, M.A. and Shabat, B.V., *Metody teorii funktsii kompleksnogo peremennogo (Methods of the Theory of Functions of a Complex Variable)*. GITTL, Moscow-Leningrad, 1935.

12. Love, A., *Matematicheskaia teoria uprugosti (Mathematical Theory of Elasticity)*. ONTI, Moscow-Leningrad, 1935.

13. Roberts, D.K. and Wells, A.A., The velocity of brittle fracture. *Engineering* Vol. 176, 1954.

14. Eshelby, J.D., Uniformly moving dislocation. *Proc. Phys. Soc.* A 62, Part 5, No. 353 A, 1949.

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